ON TAXICAB NUMBERS(6, 2, N)

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Abstract. We give solutions of a Diophantine equation $a^6 + b^6 = c^6 + d^6$ in Gaussian integers, and in numbers of the shape $x\sqrt{m} + iy$ (is there a name for such shape?)

1. Context of the problem

Is is a famous open question as to whether there are integer solutions to

$$a^6 + b^6 = c^6 + d^6. \tag{1.1}$$

There are serious heuristic considerations, according to which the equation

$$a^n + b^n = c^n + d^n \tag{1.2}$$

has no solutions in positive integers for $n \geq 5$.

However, it is known in recent times that there are Gaussian integer solutions for (1.1).

Also, we have found solutions such as the following:

$$(3 + i\sqrt{3})^6 + (1 - i\sqrt{3})^6 = (3 - i\sqrt{3})^6 + (1 + i\sqrt{3})^6 \tag{1.3}$$

$$(3\sqrt{3} + 11i)^6 + (\sqrt{3} - 11i)^6 = (3\sqrt{3} - 11i)^6 + (\sqrt{3} + 11i)^6 \tag{1.4}$$

We also consider an equation

$$a^6 + b^6 + c^6 + d^6 = 0. \tag{1.5}$$

Our goal is to find infinite sets of solutions to (1.1) and (1.5) in Gaussian integers and numbers of a type $x\sqrt{m} + iy$.

2. Solving equation $a^6 + b^6 = c^6 + d^6$ in Gaussian integers

We have an identity

$$(m(1 + i))^6 + (m(1 - i))^6 = (n(1 + i))^6 + (n(1 - i))^6 \tag{2.1}$$

for all integers m and n.

Also, more general identity

$$(a + bi)^6 + (\pm(b - ai))^6 = (c + di)^6 + (\pm(d - ci))^6 \tag{2.2}$$

In a way, the identities (2.1) and (2.2) are trivial, as both LHS and RHS of the equations are zeroes.

For the equation (1.5) we have an identity

$$(a(1 + i))^6 + (b(1 - i))^6 + (b(1 + i))^6 + (a(1 - i))^6 = 0 \tag{2.3}$$

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and

$$(a+bi)^6 + (\pm(b-ai))^6 + (c+di)^6 + (\pm(d-ci))^6 = 0 \tag{2.4}$$

3. Solving equation $a^6+b^6=c^6+d^6$ in numbers of a type $x\sqrt{m}+iy$

Let's consider an equation of the form

$$(a_0 + a\sqrt{m}i)^6 + (b_0 + b\sqrt{m}i)^6 = (c_0 + c\sqrt{m}i)^6 + (d_0 + d\sqrt{m}i))^6, \tag{3.1}$$

or equivalently,

$$(a_0\sqrt{m} + ai)^6 + (b_0\sqrt{m} + bi)^6 = (c_0\sqrt{m} + ci)^6 + (d_0\sqrt{m} + di))^6, \tag{3.2}$$

where $m > 1$ is a square-free integer.

A group of solutions is

$$(a\sqrt{3} + ai)^6 + (b\sqrt{3} + 3bi)^6 = (a\sqrt{3} - ai)^6 + (b\sqrt{3} - 3bi)^6. \tag{3.3}$$

In this identity, LHS and RHS are equal pairwise, $(a\sqrt{3} + ai)^6 = (a\sqrt{3} - ai)^6$, and $(b\sqrt{3} + 3bi)^6 = (b\sqrt{3} - 3bi)^6$, so this solution is trivial.

A few examples of non-trivial solutions:

$$(\sqrt{3} + 11i)^6 + (3\sqrt{3} - 11i)^6 = (4\sqrt{3} - 10i)^6 + (5\sqrt{3} + 7i)^6 \tag{3.4}$$

$$(\sqrt{3} + 11i)^6 + (3\sqrt{3} - 11i)^6 = (5\sqrt{3} + 7i)^6 + (7\sqrt{3} + i)^6 \tag{3.5}$$

$$(3\sqrt{3} + 11i)^6 + (5\sqrt{3} + 7i)^6 = (4\sqrt{3} + 10i)^6 + (6\sqrt{3} + 4i)^6 \tag{3.6}$$

$$(3\sqrt{3} + 11i)^6 + (5\sqrt{3} + 7i)^6 = (6\sqrt{3} + 4i)^6 + (7\sqrt{3} - i)^6 \tag{3.7}$$

$$(4\sqrt{3} + 10i)^6 + (5\sqrt{3} - 7i)^6 = (6\sqrt{3} - 4i)^6 + (7\sqrt{3} + i)^6 \tag{3.8}$$

Solutions in the form $(a_0\sqrt{m}+ai)^6+(b_0\sqrt{m}-bi)^6 = (a_0\sqrt{m}-ai)^6+(b_0\sqrt{m}+bi)^6$. As a subset of equation (3.2), we consider equation

$$(a_0\sqrt{m} + ai)^6 + (b_0\sqrt{m} - bi)^6 = (a_0\sqrt{m} - ai)^6 + (b_0\sqrt{m} + bi)^6 \tag{3.9}$$

In this form we found solutions with $m = 2, 3, 5, 15$.

We found many solutions with $m = 3$, for example:

$$(\sqrt{3} + 11i)^6 + (3\sqrt{3} - 11i)^6 = (\sqrt{3} - 11i)^6 + (3\sqrt{3} + 11i)^6 \tag{3.10}$$

$$(3\sqrt{3} + 11i)^6 + (5\sqrt{3} + 7i)^6 = (3\sqrt{3} - 11i)^6 + (5\sqrt{3} - 7i)^6 \tag{3.11}$$

Solutions with $m = 2, 5, 15$:

$$(15\sqrt{2} + 47i)^6 + (87\sqrt{2} - 71i)^6 = (15\sqrt{2} - 47i)^6 + (87\sqrt{2} + 71i)^6 \tag{3.12}$$

$$(47\sqrt{2} + 30i)^6 + (71\sqrt{2} - 174i)^6 = (47\sqrt{2} - 30i)^6 + (71\sqrt{2} + 174i)^6 \tag{3.13}$$

$$(3\sqrt{5} + 11i)^6 + (4\sqrt{5} + 3i)^6 = (3\sqrt{5} - 11i)^6 + (4\sqrt{5} - 3i)^6 \tag{3.14}$$

$$(3\sqrt{5} + 20i)^6 + (11\sqrt{5} + 15i)^6 = (3\sqrt{5} - 20i)^6 + (11\sqrt{5} - 15i)^6 \tag{3.15}$$

$$(11\sqrt{15} + 57i)^6 + (27\sqrt{15} - 61i)^6 = (11\sqrt{15} - 57i)^6 + (27\sqrt{15} + 61i)^6 \tag{3.16}$$

We found only limited number of solutions to (3.9) (and generally to (3.2)) with m different from 3, all of them listed above. It seems possible that they are the only such solutions to (3.2) (up to a common $\mathrm{GCD}(a_0, a, b_0, b)$.

On the other hand, we find many solutions with $m = 3$. It may be also possible that there is an infinite number of solutions to (3.2) with $m = 3$.

As an argument - why there are more solutions with $m = 3$, than with other m. A solution to (3.2) must satisfy

$$(3.17)\quad (-a^6 - b^6 + c^6 + d^6) + (15a^4a_0^2 + 15b^4b_0^2 - 15c^4c_0^2 - 15d^4d_0^2)m + (-15a^2a_0^4 - 15b^2b_0^4 + 15c^2c_0^4 + 15d^2d_0^4)m^2 + (a_0^6 + b_0^6 - c_0^6 - d_0^6)m^3 = 0$$

$$(3.18)\quad (6a^5a_0 + 6b^5b_0 - 6c^5c_0 - 6d^5d_0) + (-20a^3a_0^3 - 20b^3b_0^3 + 20c^3c_0^3 + 20d^3d_0^3)m + (6aa_0^5 + 6bb_0^5 - 6cc_0^5 - 6dd_0^5)m^2 = 0$$

According to (3.18), either 3 divides m, or 3 divides $(-a^3a_0^3 - b^3b_0^3 + c^3c_0^3 + d^3d_0^3)$.

We didn't find solutions of equation (1.5) of the form

$$(3.19)\qquad (a_0\sqrt{m} + ai)^6 + (b_0\sqrt{m} + bi)^6 + (c_0\sqrt{m} + ci)^6 + (d_0\sqrt{m} + di))^6 = 0.$$

However, we have a mixed-form identity

$$(3.20)\qquad (a\sqrt{m} + bi)^6 + (b - a\sqrt{m}i)^6 + (c\sqrt{n} + di)^6 + (d - c\sqrt{n}i)^6 = 0.$$

4. Open problem - Solving equation $a^6 + b^6 = c^6 + d^6$ in numbers of a type $x\sqrt{m} + y$

Let's consider an equation of the form

$$(4.1)\qquad (a_0\sqrt{m} + a)^6 + (b_0\sqrt{m} + b)^6 = (c_0\sqrt{m} + c)^6 + (d_0\sqrt{m} + d))^6,$$

where $m > 1$ is a square-free integer. It is identical to the equation (3.2), but without i. The two Diophantine equations which need to be satisfied are very similar to (3.17), (3.18), with some signs different:

$$(4.2)\quad (a^6 + b^6 - c^6 - d^6) + (15a^4a_0^2 + 15b^4b_0^2 - 15c^4c_0^2 - 15d^4d_0^2)m + (15a^2a_0^4 + 15b^2b_0^4 - 15c^2c_0^4 - 15d^2d_0^4)m^2 + (a_0^6 + b_0^6 - c_0^6 - d_0^6)m^3 = 0$$

$$(4.3)\quad (6a^5a_0 + 6b^5b_0 - 6c^5c_0 - 6d^5d_0) + (20a^3a_0^3 + 20b^3b_0^3 - 20c^3c_0^3 - 20d^3d_0^3)m + (6aa_0^5 + 6bb_0^5 - 6cc_0^5 - 6dd_0^5)m^2 = 0$$

It would be therefore reasonable to expect that the equation (4.1) has a number of solutions, similarly to the equation (3.2). We however didn't find such solutions in the same range of variables.

A question: Does equation (4.1) have non-trivial solutions?

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